# §13 Quantization of Chern-Simons Theory

In this final section of the course we want to indicate how Geometric Quantization can be applied in order to quantize a field theory. Although the field theory in question – the classical Chern-Simons field theory – is rather exotic the whole quantization procedure, which has been proposed by E. Witten in his paper on the Jones polynomial in 1989, is interesting in itself from a mathematical viewpoint as well as from a physical viewpoint. It leads to remarkable results and also to many open questions.

Chern-Simons theory is a three dimensional $(2+1)$ Yang-Mills theory. Let $M$ be a three dimensional manifold and consider the corresponding trivial principal bundle

$$P = M \times SU(N)$$

over $M$. The space $\mathcal{A}$ of gauge potentials on $P$ is an affine space whose vector space of translations is the space $\Omega^1(M,\mathfrak{g})$ of $\mathfrak{g}$-valued 1-forms on $M$ ($\mathfrak{g} = \mathrm{Lie}\, SU(N)$ is the Lie algebra of $SU(N)$). The kinematic variables are the fields $\alpha \in \mathcal{A}$ which we regard as forms $\alpha \in \Omega^1(M,\mathfrak{g})$ (after specifying a point in $\mathcal{A}$). The Lagrangian density of the CS-theory is, by definition,

$$\mathcal{L}_{CS}(\alpha) := \frac{k}{4\pi} \operatorname{tr}\left(\alpha \wedge d\alpha + \frac{2}{3}\alpha\wedge\alpha\wedge\alpha\right).$$

Here, tr is the usual trace of the matrix group $SU(N)$, $k \in \mathbb{R}$ is a constant, $d\alpha \in \Omega^2(M,\mathfrak{g})$ is the differential of $\alpha$ and $\alpha\wedge\beta$ for $\alpha,\beta \in \Omega^1(M,\mathfrak{g})$ is defined by

$$\alpha\wedge\beta\,(Y,Z) := \frac{1}{2}\left([\alpha(Y),\beta(Z)] - [\alpha(Z),\beta(Y)]\right).$$

In particular, in general, $\alpha\wedge\alpha \neq 0$. Similarly, $\alpha\wedge d\alpha$ and $\alpha\wedge\beta\wedge\gamma$ are defined.

In local coordinates we have $\alpha = \alpha_j dx^j$ and

$$\mathcal{L}_{CS}(\alpha) = \frac{k}{8\pi}\,\varepsilon^{ijk} \operatorname{tr}\left(\alpha_i(\partial_j\alpha_k - \partial_k\alpha_j) + \frac{2}{3}\alpha_i[\alpha_j,\alpha_k]\right).$$

The action functional $S$ induced by $\mathcal{L} = \mathcal{L}_{CS}$ is

$$S(\alpha) := \int_M \mathcal{L}(\alpha)$$

Note, that $\mathcal{L}_{CS}$ does not contain any metric term. Therefore, such a theory is called "topological" which means that the theory does not depend on any metric or volume. In the terminology of physics "topological" in this sense can be called also "generally covariant". From examples, in order to obtain a generally covariant theory, one introduces a metric and integrates over the space of all metrics. In this way one can think of the metric as a dynamic variable. The lesson taught by Witten and others is, that there exist highly non-trivial quantum field theories in which general covariance is realized in other ways. For example by starting with a Lagrangian density which is independent of any metric or volume.

The curvature $\Omega$ of $\alpha \in \mathcal{A}$ is the covariant derivative $D^{\alpha}\alpha = D\alpha$ of $\alpha$ and it has the form

$$\Omega = D\alpha = d\alpha + \alpha \wedge \alpha$$

in our situation.

(13.1) <u>PROPOSITION</u>: The equation of motion with respect to the action $S$ is

$$D\alpha = 0.$$

That means, that the motions of the theory, i.e. the critical points of the variation $\delta S = 0$ are the flat $SU(N)$-connections on $P = M \times SU(N)$.

☐ <u>Proof</u>. We have to solve

$$\frac{d}{d\varepsilon} S(\alpha + \varepsilon\beta)\Big|_{\varepsilon=0} = 0 \quad , \quad \varepsilon \in \mathbb{R},$$

for all $\beta \in \mathcal{A}$ (possibly with boundary conditions on $\beta$).

Since
$$\mathcal{L}_{CS}(\alpha+\varepsilon\beta) = \frac{k}{4\pi}\operatorname{tr}\Big((\alpha+\varepsilon\beta)\wedge d(\alpha+\varepsilon\beta) + \frac{2}{3}(\alpha+\varepsilon\beta)\wedge(\alpha+\varepsilon\beta)\wedge(\alpha+\varepsilon\beta)\Big)$$
$$= \frac{k}{4\pi}\operatorname{tr}\left(\alpha\wedge d\alpha + \frac{2}{3}\alpha\wedge\alpha\wedge\alpha\right)$$
$$-\varepsilon\frac{k}{4\pi}\operatorname{tr}\Big(\beta\wedge d\alpha + \alpha\wedge d\beta + \frac{2}{3}(\alpha\wedge\alpha\wedge\beta + \alpha\wedge\beta\wedge\alpha + \beta\wedge\alpha\wedge\alpha)\Big)$$
$$-\varepsilon^2\frac{k}{4\pi}\operatorname{tr}\Big(\beta\wedge d\beta + \frac{2}{3}(\alpha\wedge\beta\wedge\beta + \beta\wedge\alpha\wedge\beta + \beta\wedge\beta\wedge\alpha + \varepsilon\beta\wedge\beta\wedge\beta)\Big)$$

We conclude

$$\frac{d}{d\varepsilon}S(\alpha+\varepsilon\beta)\Big|_{\varepsilon=0} = \frac{k}{4\pi}\int_M \operatorname{tr}\Big(\beta\wedge d\alpha + \alpha\wedge d\beta + \frac{2}{3}(\alpha\wedge\alpha\wedge\beta + \alpha\wedge\beta\wedge\alpha + \beta\wedge\alpha\wedge\alpha)\Big)$$
$$= \frac{k}{4\pi}\int_M \operatorname{tr}\left(2\beta\wedge d\alpha + 2\beta\wedge\alpha\wedge\alpha\right)$$
$$= \frac{k}{2\pi}\int_M \operatorname{tr}\left(\beta\wedge D\alpha\right), \quad \text{for all suitable } \beta.$$

Here we assume $\int_M \operatorname{tr}\left(d(\beta\wedge\alpha)\right) = 0$ which follows from Stokes's theorem either by compactness of $M$, or by requiring $\beta$ to have compact support in the non-compact case (boundary condition). Hence,
$\int_M \operatorname{tr}\alpha\wedge d\beta = \int_M \operatorname{tr}\beta\wedge d\alpha$.

Therefore, $\frac{d}{d\varepsilon}S(\alpha+\varepsilon\beta)\Big|_{\varepsilon=0} = 0$ if and only if $D\alpha = 0$. □

As a result, the space of solutions is the subset $\mathcal{A}_0 \subset \mathcal{A}$ of flat connections on $P$. From the physics point of view two such solutions describe the same motion if they are related by a gauge transformation. Let

$\mathcal{G}$ be the group of gauge transformation, i.e. $\mathcal{G} \cong \mathcal{E}(M, SU(N))$ in our situation, then

$$\mathcal{M} = \mathcal{A}_0 / \mathcal{G}$$

is the true space of solutions, the space of equivalence classes of flat connections.

Note that $\mathcal{L}$ and $S$ are not completely gauge independent. Changing $\alpha \in \mathcal{A}_0$ by a $g \in \mathcal{G}$ the action picks up a constant factor $c \cdot 2\pi$. In the path integral formalism this lack of gauge invariance disappears if $c$ is an integer. This condition is a quantization condition and is related to $k$ being an integer as well. We assume for the following that $k$ is a positive integer.

The above phenomenon is related to $\pi_3(SU(N)) \cong \mathbb{Z}$. Because of this property the group $\mathcal{G}$ of differentiable maps $M \to SU(N)$ is not connected and we have a proper gauge invariance only for the connected component $\mathcal{G}_1 \subset \mathcal{G}$ of the identity.

In general, the whole classical CS-theory can be carried through for an arbitrary simple, compact, simply connected Lie group $G$ instead of $SU(N)$.

So far we have arrived at a classical phase space, the space $\mathcal{M} = \mathcal{A}_0/\mathcal{G}$ of equivalence classes of flat $SU(N)$-connections on $M$. To come to a quantum model we need to specify a class of gauge invariant observables. The usual functions or operators are not generally covariant or they are trivial. But the so called Wilson lines - familiar in QCD - give a natural set of gauge invariant observables.

For example, let $\alpha$ be a flat connection and $\gamma$ an (oriented) closed curve in $M$. Let $\varrho$ be an irreducible representation of $G = SU(N)$. Computing the holonomy of $\alpha$ around $\gamma$ we get an element of $SU(N)$ (up to conjugacy): to $a \in \gamma$ we consider the parallel transport of $p \in P_a$ along $\gamma$ ending up with $pg \in P_a$, $g \in SU(N)$. $g$ is unique up to conjugacy and the observable "Wilson line" is

$$W_{\varrho,\gamma}(\alpha) = \mathrm{tr}_\varrho\, g$$

which can also written as

$$W_{\varrho,\gamma}(\alpha) = \mathrm{tr}_\varrho \left( P \exp \int_\gamma \alpha \right),$$

where "P" stands for "path ordered" integration. For the incorporation of knots and links into the Chern-Simons theory we get more observables by taking $m$ oriented and non-intersecting closed curves $\gamma_j$, $j=1,\dots,m$, i.e. knots, and irreducible representations $\varrho_1,\dots\varrho_m$ in order to arrive at the Feynman path integral

$$\int_{\mathcal{M}} \prod_{j=1}^{m} W_{\varrho_j,\gamma_j}(\alpha) \exp(iS(\alpha))\, D\alpha .$$

This can be viewed (if mathematically well-defined) as the "partition function" $Z(M) = Z(M,\varrho,\gamma)$ of $M$ with respect to $\varrho_j, \gamma_j$. We do not pursue this line of arguments but rather exploit the idea of the Wilson lines to formulate the following result:

(13.2) <u>PROPOSITION</u>: Parallel transport defines a bijection

$$\mathcal{M} = \mathcal{A}_0/\mathcal{G} \longrightarrow \mathrm{Hom}(\pi_1(M), G)/_G .$$

□ <u>Sketch</u> of proof: Fixing $a \in M$ we consider the fundamental group $\pi_1(M)$ as the quotient of the full group of differentiable and closed curves starting and ending in $a$. For each $\alpha \in \mathcal{A}_0$ the parallel transport with respect to $\alpha$ along such a curve $\gamma$ picks a $g = g(\alpha,\gamma) \in SU(N)$, so that $pg \in P_a$ is the parallel transport of $p \in P_a$. Since $\alpha$ is flat the parallel transport is locally independent of the paths. Hence, the element

$g(\alpha,\gamma)$ only depends on the homotopy class of $\gamma$ in $\pi_1(M)$. Moreover, $g(\alpha,\gamma)$ is independent of the choice of $\alpha$ within a class $\in \mathcal{M} = \mathcal{A}_0/\mathcal{G}$ and independent of the choice of the point $a$ up to conjugacy. Clearly, for a fixed $\alpha \in \mathcal{A}_0$ the map $\gamma \longmapsto g(\alpha,\gamma)$ is a homomorphism and descends to a homomorphism

$$\pi_1(M) \longrightarrow SU(N).$$

Altogether, we get a map

$$\mathcal{M} = \mathcal{A}_0/\mathcal{G} \longrightarrow \mathrm{Hom}(\pi_1(M), G)/_G$$

which is certainly injective.
To show the surjectivity one constructs for a given homomorphism $R : \pi_1(M) \to G$ a suitable [*] bundle with locally constant transition functions (a local system) which leads to the appropriate flat connection $\alpha$ inducing the given homomorphism $R$.

With the identification $\mathcal{M} \cong \mathrm{Hom}(\pi_1(M), G)/_G$) we now can compare $\mathcal{M}$ with other natural constructions. For example, if we extend the Čech cohomology to nonabelian groups $G$ then we see a natural identification of $\check{H}^1(M,G)$ and $\mathrm{Hom}(\pi_1(M), G)/_G$. Moreover, $\check{H}^1(M,G)$ can be identified with the group

[*] Construct the connection!

cohomology $H^1(\pi_1(M), G)$.

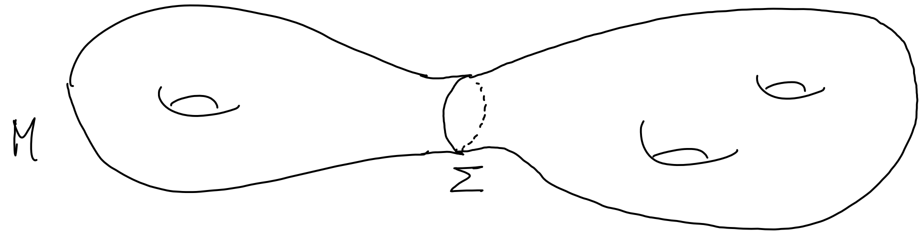
Note, that the identifications above

$$\boxed{\mathcal{M} = \mathcal{A}_0/\mathcal{G} \cong \mathrm{Hom}(\pi_1(M), G)/G \cong \check{H}^1(M, G) \cong H^1(\pi_1(M), G)}$$

are independent of the dimension of M. In addition, in the dimension $\dim M = 2$ one has an identification of $\mathcal{M}$ with a moduli space of stable holomorphic bundles over M (when M is a compact and connected two manifold ("a surface") and when M is equipped with a complex structure $J$, i.e. M with $J$ is a Riemann surface). This moduli space $\mathcal{M}_J$ is a complex analytic space whose non-singular part is a Kähler manifold. This is the point where Geometric Quantization enters the stage!

Before we arrive at this application we want to explain the quantization strategy which is influenced by Feynman integral manipulations.

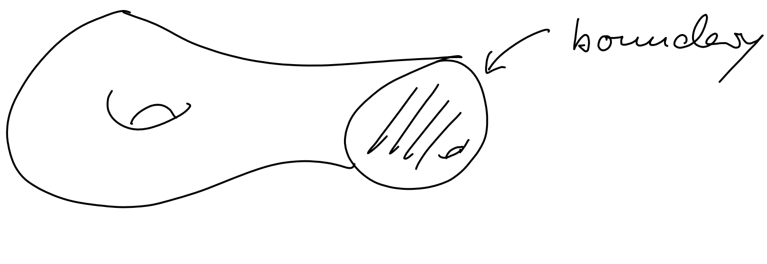
The basic strategy for solving the Yang-Mills theory with CS-action on a three dimensional manifold M (at the quantum level) is to develop a theory for chopping M into pieces solve the problem for the pieces and then glueing the results to obtain a solution for M. We cut the manifold M along a compact surface $\Sigma$:




Near $\Sigma$, $M$ looks like $\Sigma \times I$ with an open intervall $I \subset \mathbb{R}$. So we first have to understand the case of a cylinder $M \cong \Sigma \times I$!

Canonical quantization on $\Sigma \times I$ will produce the quantum Hilbert space $\mathcal{H} = \mathcal{H}_\Sigma$ ($= Z(\Sigma)$ in the notation below) which turns out to be finite dimensional. Similar spaces have been constructed in conformal field theories or with respect to generalized thetafunctions.

A precise formulation of the chopping and glueing "theory" leads to the concept of topological field theory TQFT (or topological quantum field theory). The main ingredient of such a TQFT is the following: The class of manifolds considered is extended to all oriented, compact three manifolds $M$ with boundary, e.g.



To each surface $\Sigma$ (i.e. compact, oriented 2-dim. manifold without boundary) we attach (or want to construct) a finite dimensional vector space over $\mathbb{C}$

$$Z(\Sigma).$$

And to each three dimensional manifold $M$ with bdry $\partial M$ – possibly $\emptyset$ – we attach a vector

$$Z(M) \in Z(\partial M),$$

such that several compatibility properties and norming conditions are satisfied.
For example, $\emptyset = \Sigma$ as a surface yields

$$Z(\emptyset) = \mathbb{C},$$

and $\emptyset = M$ as three manifold with $\partial\emptyset = \emptyset$ yields

$$Z(\emptyset) = 1 \in Z(\emptyset) = M.$$

In general, if $M$ is a manifold without boundary, i.e. $\partial M = \emptyset$, we get a topological invariant

$$Z(M) \in Z(\emptyset) = \mathbb{C}.$$

If $\Sigma^*$ denotes the surface $\Sigma$ with the opposite orientation we require

$$Z(\Sigma^*) = Z(\Sigma)^*$$

where $V^*$ is the dual of the complex vector space $V$.

For the disjoint union of surfaces $\Sigma = \Sigma_1 \cup \Sigma_2$ one requires $Z(\Sigma) = Z(\Sigma_1) \otimes Z(\Sigma_2)$ (like a multiplicative) homology; and correspondingly for the disjoint union $M_1 \cup M_2$ of 2 three manifolds one requires $Z(M) = Z(M_1) \otimes Z(M_2)$, hence $Z(M) \in Z(\partial M_1) \otimes Z(\partial M_2) = Z(\partial M)$.

If $M = M_1 \cup M_2$ with $\partial M = \phi$ and $M_1 \cap M_2 = \Sigma$ a surface, see above sketch, then $\partial M_1 = \Sigma^*$ if $\partial M_2 = \Sigma$ and the number $Z(M) \in \mathbb{C}$ is given by evaluation

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle = Z(M_1)\,(Z(M_2)) \in \mathbb{C}$$

(Note, that $Z(M_2) \in Z(\Sigma)$ and $Z(M_1) \in Z(\Sigma^*) = Z(\Sigma)^*$.)
Last not least, everything should be invariant under diffeomorphisms and more.

Without dealing with TQFT's we concentrate, for the rest of the lesson on the construction of $Z(\Sigma) = \mathcal{H}_\Sigma$ for a given surface $\Sigma$.

Coming back to the "cylinder" $\Sigma \times I$ we can use a suitable gauge: $\alpha$ vanishes in the "I-direction" (in coordinates $q^0, q^1, q^2$, where $q^0$ denotes the direction $I$, and $\alpha = \alpha_j dq^j$ the gauge is $\alpha_0 = 0$).
In this gauge the arguments carry over and we have to quantize the classical phase space

$$\mathcal{M} = H^1(\Sigma, SU(N)) \quad \left(\text{or} \quad H^1(\Sigma, G)\right).$$

The case $G = \mathcal{U}(1)$:

As an illustrating example we first study the case of the internal gauge group $\mathcal{U}(1)$. In this case we have an abelian Yang-Mills theory.

Our space of equivalence classes of flat connections is

$$\mathcal{M} = H^1(\Sigma, \mathcal{U}(1))$$

where $\Sigma$ is, as before, an oriented, compact two dim. manifold of genus $g$, $g = 0, 1, \dots$
The quotient map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \cong \mathcal{U}(1)$ leads to a natural representation of $H^1(\Sigma, \mathcal{U}(1))$ as a quotient: With respect to the integer lattice

$$\Lambda := H^1(\Sigma, \mathbb{Z}) \subset H^1(\Sigma, \mathbb{R}) \cong \mathbb{R}^{2g}$$

we have the interpretation of $\mathcal{M}$ as the torus

$$\mathcal{M} = H^1(\Sigma, \mathcal{U}(1)) \cong H^1(\Sigma, \mathbb{R})/_{\Lambda} \cong \mathbb{R}^{2g}/_{\Lambda}.$$

Now, $H^1(\Sigma, \mathbb{R})$ has a natural symplectic form $\omega$ induced by

$$(\alpha, \beta) \longmapsto \int_\Sigma \alpha \wedge \beta$$

for one forms $\alpha, \beta$ on $\Sigma$. And this form descends to the torus $H^1(\Sigma, \mathbb{R})/_{\Lambda}$.

Without additional choices there is no natural way to quantize this torus $(H^1(\Sigma, U(1)), \omega)$.

The situation would be better, if we would have a Kähler structure in order to be able to carry out the geometric quantization procedure with respect to the holomorphic polarization.

There is a natural way to get a Kähler structure on $H^1(\Sigma, U(1))$ coming from function theory! Indeed, let $\tau$ be a complex structure on $\Sigma$ then we have

$$H^1(M, \mathbb{R}) = H^1(M, \mathcal{O}),$$

which also can be regarded as the space of holomorphic one forms (by Serre duality) and hence carrying a natural complex structure $\mathcal{J}$ ( $\mathcal{J} : H^1(M,\mathbb{R}) \to H^1(M,\mathbb{R})$ with $\mathcal{J}^2 = -1$).

$\mathcal{M}$ equipped with the complex structure induced by $\tau$ will be denoted by $\mathcal{M}_\tau$. $\mathcal{M}_\tau$ is in fact the Jacobian of $\Sigma_\tau$. By function theory we know that $\mathcal{M}_\tau$ has a natural holomorphic line bundle $L_\tau$, the theta bundle, with compatible connection such that $\mathrm{Curv}(L, \nabla) = \omega$. Quantizing our system means to take

$$Z(\Sigma_\tau) = \Gamma_{hol}(\mathcal{M}_\tau, L_\tau)$$

as the quantum Hilbert space. It is well-known that $\dim_{\mathbb{C}} \Gamma_{hol}(\mathcal{M}_\tau, L_\tau) = 1$ and that $\Gamma_{hol}(\mathcal{M}_\tau, L_\tau)$ is generated by the thetafunction attached to $\Sigma_\tau$ (resp. $\mathcal{M}_\tau$).

Varying our original parameter $k$ (so far $k=1$) means to take $L_\tau^k = L_\tau \otimes \dots \otimes L_\tau$ as the quantum line bundle with conn. $\nabla^k$ satisfying $\mathrm{Curv}(L_\tau^k, \nabla^k) = k\omega$. The corresponding quantum Hilbert space is

$$Z^k(\Sigma_\tau) = \Gamma_{hol}(\mathcal{M}_\tau, L_\tau^k).$$

$Z^k(\Sigma_\tau)$ is finite dimensional, since $\mathcal{M}_\tau$ is compact, and by Riemann-Roch it can be shown that

$$\dim Z^k(\Sigma_\tau) = k^g,$$

independently of the complex structure.

We cannot be content since our result, the space $Z^k(\Sigma_\tau)$, depends on $\tau$. This contradicts the topological (generally covariant) nature of all the ingredients of a TQFT. The only way to solve this problem is to naturally identify the various $Z^k(\Sigma_\tau)$, $\tau \in \mathcal{J}$ = "Teichmüller space".

$\mathcal{T} = \mathcal{T}_g$ is the space of complex structures on a given surface $\Sigma$. $\mathcal{T}$ is itself a complex manifold and simply connected. For example, in the case of $g=0$ we have $\mathcal{T} = \{*\}$ and for $g=1$ one knows $\mathcal{T}_1 \cong \mathbb{C}$ (recall that all Riemann surfaces of genus $g$ are of the form $\mathbb{C}/\Lambda(\tau)$ where $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$, and $\Lambda(\tau) = \{ \ell + m\tau : \ell, m \in \mathbb{Z} \}$.)

Our vector spaces $Z^k(\Sigma_\tau)$ form a vector bundle $Z^k$ over $\mathcal{T}_g$ which has a natural holomorphic structure. We know already that all $Z^k(\Sigma_\tau)$ are isomorphic since they have the same dimension. But we want more, we want to naturally identify them. That is we want to have a natural choice of the necessary isomorphisms.

Assume that we have a natural flat connection on the vector bundle $Z^k \to \mathcal{T}_g$. Then parallel transport is independent of the curves since $\mathcal{T}_g$ is simply connected and we obtain our natural isomorphisms.

(13.3) <u>PROPOSITION</u>: There is a natural connection on $Z^k \to \mathcal{T}_g$ which is projectively flat.

Thus all the $\mathbb{P}(Z^k(\Sigma_\tau))$ are naturally identified which is all we need from the physical point of view.

Note, that there is a subtlety in the argument, since some of the complex structures $\tau$ occur several times in $\mathcal{T}$. Therefore, one should deal with the true moduli space $\mathcal{M}_g = \mathcal{T}_g / \Gamma_g$.

<u>The case $SU(N)$, $N \geq 2$</u>

After this baby case with $G = U(1)$ we now sketch the nonabelian case $G = SU(N)$.

On $\mathcal{M} = H^1(\Sigma, SU(N))$ we first fix a topological structure (the quotient topology on $\mathcal{A}_0/\mathcal{G}$):
$\pi_1(\Sigma)$ is generated by $A_1, \dots A_g, B_1, \dots B_g$ (we consider $g \geq 2$ only) with the only relation

$$\prod_{j=1}^{g} [A_j, B_j] = 1$$

Set $V := \{ (z,w) \in G^g \times G^g \mid \prod_{j=1}^{g} [z_j, w_j] = 1 \}$. $V$ is closed in $G^g \times G^g$, hence $V$ obtains a compact topology, and the same is true for $V/G$. $V/G$ has an open and dense subset where $V/G = \mathcal{M}$ is a manifold: If for $R \in \mathcal{M}$, $R(\pi_1(\Sigma))$ generates $G$, $R$ is a regular point. (Here we need $g \geq 2$, since in the cases $g=0, g=1$ the group $G$ is not generated by $R(\pi_1(\Sigma))$.)

Let us denote $\mathcal{M}^s$ to be the manifold points of $\mathcal{M}$. On $\mathcal{M}^s$ there is a natural symplectic structure which can by obtained by again integrating $\int_\Sigma tr(\alpha \wedge \beta)$ for $\alpha, \beta$ one forms on $\Sigma$.

But no natural quantization is available without additional structure.

If we pick again a complex structure $\tau$ on $\Sigma$ then the holomorphic vector bundles $E$ over $\Sigma_\tau$ of rank $N$ $(G = SU(N))$ are important.
According to a theorem of Narasimhan and Seshadri (1965) on the topological space $\mathcal{M}$ there exists a structure of a complex analytic space (cpx manifold with singularities) which is the moduli space $\mathcal{M}_\tau$ of semistable holomorphic vector bundles of rank $N$ and vanishing Chern class. This is a deep theorem with many generalisations. The result also contains the proposition, that the manifold part of $\mathcal{M}_\tau$ (as moduli space) is a Kähler manifold with respect to the symplectic form considered above.

Moreover, $\mathcal{M}_\tau$ has a unique holomorphic line bundle (the theta bundle) $L_\tau$ with compatible connection $\nabla$ such that $\mathrm{Curv}(L_\tau, \nabla) = \omega$ on the nonsingular part. Thus we obtain the quantum Hilbert space

$$Z(\Sigma_\tau) := \Gamma_{hol}(\mathcal{M}_\tau, L_\tau) \qquad (= \mathcal{H}_P\,,\ P \text{ hol. polarization})$$

and correspondingly for $k \in \mathbb{N}$:

$$\mathrm{Curv}(L_\tau^k, \nabla^k) = k\omega \quad \text{with}$$

$$Z^k(\Sigma_\tau) := \Gamma_{hol}(\mathcal{M}_\tau, L_\tau^k).$$

In our situation ( $SU(N)$ as symmetry group) we have $L_\tau^{-N} = S$ as a square root of the canonical line bundle $K$ on $\mathcal{M}_\tau$. As a result the metaplectic correction yield

$$Z^k(\Sigma_\tau) := \Gamma_{hol}(\mathcal{M}_\tau, L_\tau^{k-N}).$$

All these space are finite dimentional and for fixed $k$ the dimentions are constant. Moreover, as in the case of $G = U(1)$, the spaces $Z^k(\Sigma_\tau)$ form a holomorphic vector bundle

$$Z^k \longrightarrow \mathcal{T}_g .$$

It is another deep theorem (of Hitchin (1990)) that there exists a natural projectively flat connection on these bundles which identify there fibres $Z_\tau^k$ as projective spaces.

We therefore have construkted a vector spase

$$Z^k(M) \left( := Z_\tau^k(M) \right),$$

which is independent of the chosen complex structure $\tau \in J_g$ up to a constant, i.e $\mathbb{P}(Z^k(M))$ is independent of $\tau \in J_g$.

The program of TQFT now requires to construct the $Z(M) \in Z(\partial M)$ of the theory and the application to knots requires even much more (open till today), namely the study of the moduli spaces $\mathcal{M}(\Sigma \setminus D, SU(N))$, where $D \subset \Sigma$ is a finite set of points (i.e. the support of a divisor $D$ on $\Sigma$).

Our purpose was to show how geometric quantization is applied to Chern-Simons theory on 3 manifolds in the absence of knots or links.

Literatur zum Thema Chern-Simons-Theorie erscheint in 'Bibliography'.